Introduction to General Relativity: Test

Time: 1.5 hours, Total marks: 50

1. By considering the flux of particle number across the surface of a cube of side a and letting $a \to 0$, derive the conservation law:

$$N^{\alpha}{}_{,\alpha} = 0 \ .$$

where $N^{\alpha} = n(c, \mathbf{v})$ is the number flux 4-vector.

[10 marks]

2. $F^{\alpha\beta}$ are the components of an anti-symmetric tensor and $T_{\alpha\beta}$ are the components of a symmetric tensor. Prove that $F^{\alpha\beta}T_{\alpha\beta} = 0$.

[5 marks]

3. Show that the components of the derivative of a vector $V^{\alpha}{}_{,\beta}$ do not transform like a tensor. For what class of coordinate transformations are $V^{\alpha}{}_{,\beta}$ the components of a $1-1$ tensor.

[10 marks]

4. A particle moves through the lab with a 3 - velocity $\mathbf{v}$.

   (a) Write down the 4 - velocity of a particle in its own rest frame.

   (b) By performing a Lorentz transformation to the lab, calculate the components of the 4 - velocity measured by an observer in the lab.

[2+3 marks]

5. In a frame $\mathcal{S}$, a photon has a frequency ν and moves at an angle Θ with respect to the x - axis.

   (a) Write down the 4 - momentum of the photon.

   (b) By performing a Lorentz transformation, find the frequency in frame $\overline{\mathcal{S}}$ when $\Theta = \pi/6$.

   (c) What happens when $\Theta = \pi/2$ and what is the significance of this result.

   (d) A taxi towards a red traffic light. How fast would it have to go for it to appear green [$\nu_{red} = 4 \times 10^{14} s^{-1}$, $\nu_{green} = 6 \times 10^{14} s^{-1}$].

[2+3+2+3 marks]

6. An atom at rest in a laboratory emits a photon and recoils. If its initial rest mass is m_0 and it loses rest energy e in the emission.

   (a) Use conservation of 4 - momentum or otherwise to prove that the frequency of the emitted photon is given by

$$\nu = \frac{e}{h}\left(1 - \frac{e}{2m_0c^2}\right) \ .$$

   (b) Find the velocity of recoil.

[5+5 marks]